Set 1

1. (3) Compute the number of positive integers k such that $\frac{2020-k}{k}$ is also a positive integer.

2. (3) Solve for x:
$$(2+0+2+0)x = 2020$$

3. (3) Two nonzero real numbers a and b satisfy $\frac{a}{b} = 2020$. Compute all possible values of $\frac{\min(a,b)}{\max(a,b)}$.

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Set 2

1. (4) The Kansas City Chiefs played a football game against the Toronto Pigeons! Whenever the Chiefs started a drive, the drive ended in one of two ways:

   - Regular touchdown, worth 7 points.
   - Touchdown with two-point conversion, worth 8 points.

   Overall, the Chiefs scored T touchdowns and earned 100 points from them (while the Pigeons did not score at all). Compute the number of possible values of T.

2. (4) Ana and Banana are setting problems for a math contest. Working alone, Ana can create one problem in 48 minutes. However, when she works together with Banana, they both work at twice their regular speeds. This allows them to create one problem in only 8 minutes. Given that both Ana and Banana create problems at constant rates, how many minutes does Banana when working alone need to create one problem?

3. (4) A number from 1 to 200, inclusive, is chosen. What is the probability that it can be expressed as a difference of two non-negative squares?

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Set 3

1. (5) Let ABC be a triangle. Let M be the midpoint of BC, and let P and Q be on AC and AB respectively such that $\frac{AP}{PC} = \frac{BQ}{QA} = 2$. If D is the midpoint of BP, E is the midpoint of CQ, F is the midpoint of MP, and G is the midpoint of QM, find the ratio between the areas of quadrilateral $FEGD$ and triangle ABC.

2. (5) Let $a_1, a_2, \ldots$ be a sequence recursively defined as:

   (a) $a_1 = 1$

   (b) $a_2 = 3$

   (c) $a_k = 4a_{k-1} - 4a_{k-2} + 1$

   Compute a_{11}.

3. (5) Let k be a positive integer in base 4, and let $\#\#\#_4$ denote a numeral in base 4. We wish to reduce k to 0 by repeating the following procedure: Select any two nonzero digits of k (in base 4), and reduce each of those digits by 1. For how many positive integers $k \leq 1000_4$ is this possible? (For example, for $k = 222_4$, we can do $2\underline{22} \to \underline{2}1\underline{1} \to \underline{11}0 \to 0$.)

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Set 4

1. (5) Consider an isosceles right triangle AOB such that $AO = BO$. Say A, O, and B all lie on a semicircle such that AB is the base of this semicircle. Consider points C and D on the semicircle such that AC bisects $\angle OAB$ and BD bisects $\angle OBA$. Finally, denote X as the intersection of AC and BD. Find $\angle CXD$.

2. (5) Let $ABCD$ be a rectangle such that $AB = 3$ and $BC = 4$. Consider a point P on line BD such that the centroid of APC lies on BC Find the area of triangle APC.

3. (5) Michael is trying to roll a 6 with a standard six-sided die. However, before each roll, there is an independent probability of p that he gets distracted and never rolls the die again. If he has a $\frac{5}{6}$ chance of eventually rolling a 6, compute the value of p.

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Set 5

1. (6) Consider a triangle ABC where $AB < AC$. Let I be its incenter and let K be a point on the circumcircle of ABC such that KB is perpendicular to AI. Let KC intersect the circumcircle of BIC again at M. If M is the A-excenter and $\angle ABC = 40°$, find $\angle ACB$.

2. (6) Simhon wins a game with probability p between 0 and 1, inclusive. If he wins, he will get $(3 - 3p)$ dollars, but if he loses he will lose $(p+1)$ dollars. Compute the value of p that maximizes the expected value of his profit.

3. (6) A *harmonic sequence* $a_1, a_2, \ldots$ is defined such that

$$\frac{a_2 - a_1}{a_1} = \frac{a_3 - a_2}{a_3}$$
$$\frac{a_3 - a_2}{a_2} = \frac{a_4 - a_3}{a_4}$$

and so on, where all members of the sequence are nonnegative.

Consider a harmonic sequence that satisfies that a_1 is an integer, $a_2 = 10$, and the sequence is strictly increasing (so no two terms are equal). Given that this harmonic sequence is not infinitely long, find the largest possible length of the sequence.

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Set 6

1. (6) You receive the following problem during a UTMC Individual round:

> Let r_1 and r_2 be the two distinct nonzero roots of the cubic equation $Q(x) = x^3 - 20yx^2 + 15yx = 0$. Compute the value of $\frac{1}{r_1} + \frac{1}{r_2}$.

   Interestingly, the question never gives you the value of y. Compute the value of $\frac{1}{r_1} + \frac{1}{r_2}$ anyway, and express it as a real number not in terms of y.

2. (6) Let A be a point on a circle, and let B and C be two different points on this circle such that $AB = AC \neq 0$. Consider two points D and E on the circumcircle of ABC such that AD and BE are parallel and AE and CD are parallel. Finally, denote X as the intersection of BD and CE. If $AX = DE$, and A, B, E, D, and C lie on the circle in this order, compute $\angle DAE$.

3. (6) The UTMC Committee loves making problems, but such hard work must come at a cost! Quite literally, as the nth problem on the contest consumes $\frac{n^2+1}{n}$ dollars. If the committee begins with 2020 dollars, and cannot create a problem unless it can pay the full cost of producing it, what's the maximum number of problems that can they make?

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Set 7

1. (7) a, b, and c are the (possibly complex) roots of the polynomial

$$x^3 - 6x^2 + 12x - 9$$

If $|a| \geq |b| \geq |c|$, find $|a(b+c)|$.

2. (7) Let $f(n)$ denote the number of positive factors of an integer n. Find the number of integers n between 1 and 2019 inclusive such that $f(n)$ is an odd prime.

3. (7) Let ABC be a triangle with $AB = 32$, $AC = 50$, $BC = 45$. Let ω be a circle passing through A and tangent to BC at T, that intersects AB at D and AC at E. If $BD = CE$, compute BT.

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Set 8

1. (8) Compute the maximum possible remainder when $86^n - 30^n + 21^n - 5^n$ is divided by 216 over all nonnegative integers n.

2. (8) Let $ABCD$ be a cyclic quadrilateral with circumcenter O, such that $BC = CD = 2$ and BD bisects AC. If $AC = BO$, find the area of quadrilateral $ABCD$.

3. (8) The function $f : \mathbb{Q} \to \mathbb{Q}$ satisfies for all nonzero rationals x, y the functional equation:

$$f(x) - f\left(\frac{1}{y}\right) = \frac{xf(y) - f(1)}{y}$$

Compute the number of functions f such that $0 < f(10) < 2020$ and $f(3)$ is an integer.

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Set 9

1. (9) Compute:

$$\sum_{a=1}^{8}\sum_{b=1}^{8}\frac{a^3+b}{a+b}$$

2. (9) Let a_n be an infinite sequence of numbers satisfying $a_0 = 2019$ and $a_{n+1} = 10^8 \cdot a_n + 20202019$. Compute the number of digits, in base 10, of the first number in this sequence that is divisible by 99.

3. (9) Ethan Liuwu is trapped in a dungeon! He is inside a dungeon corridor, which consists of 7 tiles in a row, numbered $1, 2, \ldots, 7$ from left to right. Unfortunately, the dungeon is pitch dark and thus Ethan Liuwu moves at random. Every second, he has a $\frac{1}{2}$ chance of moving one tile to the left, a $\frac{1}{4}$ chance of moving one tile to the right, and a $\frac{1}{4}$ chance of staying on the same tile. If he reaches tile 1, he will safely leave the dungeon, never to return. However, if he reaches tile 7, he'll fall into a bottomless pit and get trapped forever. Compute the probability that Ethan Liuwu will leave the dungeon safely, given that he starts on tile 6.

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1. (10) Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. Let the internal angle bisector of $\angle BAC$ intersect the circumcircle of triangle ABC at M, and let the tangents from M to the incircle of ABC meet the circumcircle of triangle ABC at P and Q (both of which are not point M). Find the length of PQ.

2. (10) Find the smallest integer $x > 11$ such that there exists an integral n such that:

$$S = \frac{\log_{x-10}(n-10)}{\log_x n}$$

is a rational number greater than 1.

3. (10) Find the probability that for any positive integer n, we have that 61 divides:

$$20^n - 210^n + 70^n - 125^n + 245^n$$

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