UTMC Senior Individual Round

1. (4) What is the maximum value of $a^{b^{c^d}}$, given that a, b, c, d are all distinct integers chosen from the set $(0, 1, 2, 9)$?

2. (5) Define the operation $\diamond$ such that:
$$a \diamond b = ab - 9a + 9$$
Compute:
$$(\dots((1 \diamond 2) \diamond 3) \dots \diamond 9) \diamond 10$$

3. (6) Let d be a randomly selected positive (not necessarily proper) divisor of 6^6. Compute the expected value of the number of positive divisors of d.

4. (6) For non-zero real numbers m and n, define the operation $\bigstar$ as $m \bigstar n = \frac{1}{m} + \frac{1}{n} - \frac{1}{mn}$. Compute the value of
$$4 \bigstar (5 \bigstar (\dots (2018 \bigstar 2019) \dots))$$

5. (6) Given that there exists four primes for which the sum of any three is a prime number, find the minimum possible value of the smallest prime out of the four.

6. (6) Let $ABCDEF$ be an convex hexagon. Let points G, H, I, J, K, and L be the midpoints of AB, BC, CD, DE, EF, and FA, respectively. Let point P be a point on the interior of the hexagon such that the areas of quadrilaterals $BGPH$, $CHPI$, $DIPJ$, $EJPK$, and $FKPL$ are 2018, 2019, 2020, 2021, and 2022, respectively. Compute the area of hexagon $ABCDEF$.

7. (7) The company MMM (Make More Money) made a whopping profit of \$20 last year, and the N owners are splitting the money using the following process. The youngest owner proposes a plan where each owner receives a positive integer number of dollars, such that the youngest owner maximizes their own profit while still satisfying the other owners. After the split, each owner votes. (Assume that all owners have different ages.) If at most 1 owner votes against the plan, it passes; otherwise, the youngest owner gets fired and receives \$0, and the process repeats for the remaining $N - 1$ owners. Each owner tries to maximize the amount of money they receive; if there are multiple possible moves that do this, they try to fire as many owners as possible. Assume that the owners do not collude, but that they act optimally otherwise. Given that $N = 4$, compute the amount of money that the youngest owner receives.

8. (7) Let ABC be a triangle satisfying $AB = 3$, $AC = 5$, $BC = 7$. Let A' be the reflection of A across BC, and let the tangents to the circumcircle of triangle ABC from points B and C intersect at a point K. Find $A'K$.

9. (8) Michael is playing a game with his 4 friends to split a single cake. To begin, each player flips one coin. Then, all people who flip "Heads" share the cake equally. If everyone flips "Tails", the cake is thrown out and nobody gets to eat it. What is the expected amount of cake that Michael will get?

10. (8) Let x be a real number such that $\log_4(x), \log_8(4x),$ and $2020 + \log_{32}(x)$ are three consecutive terms in a geometric sequence (in that order). The product of all possible values of x can be written in the form 2^n for some real number n. Compute n.

11. (9) a and b are one digit positive integers in base k. If $2a \cdot b$ represented in base k forms the two digit number ab, find the number of possible values of integers k from 2 to 2020 inclusive. Here, we say that the two digit number ab is the number formed by sticking the digits a and b in base k together in that order: for example, if $k = 10$, $a = 3$, and $b = 5$, then this number would be 35.

12. (10) Let ABC be a triangle with $AB = 5$, $AC = 12$, and $BC = 13$. Let the incircle of ABC touch AB and AC at F and E respectively, and let the line EF intersect the circumcircle of triangle ABC at two points P and Q. Compute the length of PQ.